Slide 0

Fuzzy Logic

오늘의 교훈

미래는 준비하는자의 것이다

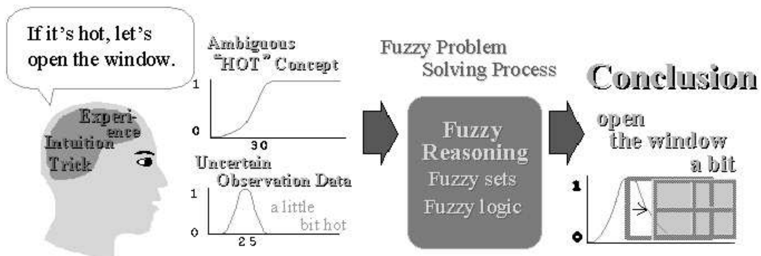
Fuzzy Logic

Slide 1

- What is fuzzy logic?
  - a superset of conventional (Boolean) logic
  - extend to handle the concept of partial truth—truth values between "Completely true" and "Completely false"
  - introduced by Dr. Lotfi Zadeh of UC/Berkeley in the 1960's
  - a means to model the uncertainty of real world
  - approximate reasoning methods to produce a relatively clear answer
  - A Fuzzy set is a set having vague boundaries.
  - Fuzzy sets can successfully represent a human's ambiguous estimations.
  - The fuzzy set is represented by a "membership function" which takes values between 0.0 and 1.0.
  - Fuzzy reasoning uses this membership function to describe the human's reasoning process successfully.

Fuzzy Logic (Cont'd)

Slide 2

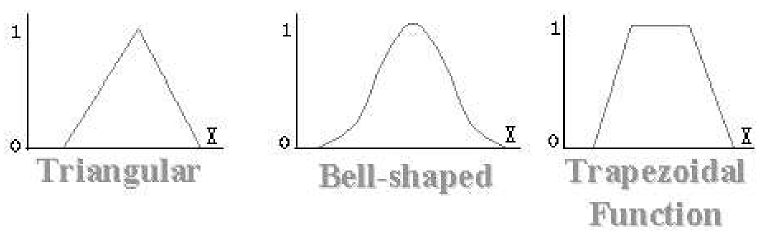


- History
  - in 1965, Zadeh introduced fuzzy set theory
  - in 1974, Mamdani applied to fuzzy control
  - in 1980, first industrial application was reported
  - in 1987, many applications of fuzzy logic control were published

Fuzzy Logic (Cont'd)

Slide 3

- Fuzzy Set Operations
  - Fuzzy logic is based on the concept of fuzzy set.
  - Fuzzy set have manipulated the idea of the human's ambiguous estimation in a mathematical way.
  - Fuzzy set is defined by membership function.
  - The shape of the various membership functions are selected by their fuzzy set characteristics.
  - Some frequently used membership functions are shown here.



  - For instance, when we say [HOT], it has lots of ambiguous and subjective

Fuzzy Logic (Cont'd)

estimation.

– (i.e. "Its Hot, it must be hotter than yesterday! How can you say you are cold?)

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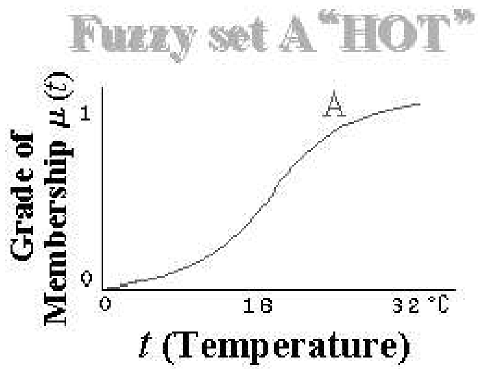

Fuzzy Logic (Cont'd)

– The basic operations on fuzzy sets consist of three logical operators; AND, OR, NOT.

(1) Fuzzy Set Complement "NOT": For instance, what is a complement of fuzzy set A, "hot" room-temperature?

∗ The fuzzy set complement A^c is defined by the equation; $\mu_{A^c}(t) = 1 - \mu_A(t)$

∗ This is a fuzzy set of [COLD].

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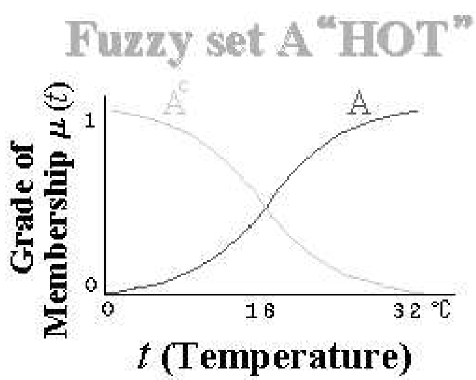

Fuzzy Logic (Cont'd)

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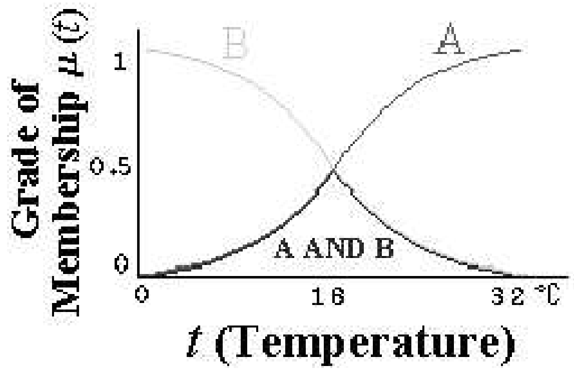
(2) Fuzzy Set Intersection "AND":

* The fuzzy set of [NOT COLD AND NOT HOT] is the area where the grade of membership for both $B(= A^c)$ [COLD] and A [HOT] fuzzy sets:
$AANDB = \mu_A(t) \wedge \mu_B(t)(\wedge : min)$



Fuzzy Logic (Cont'd)

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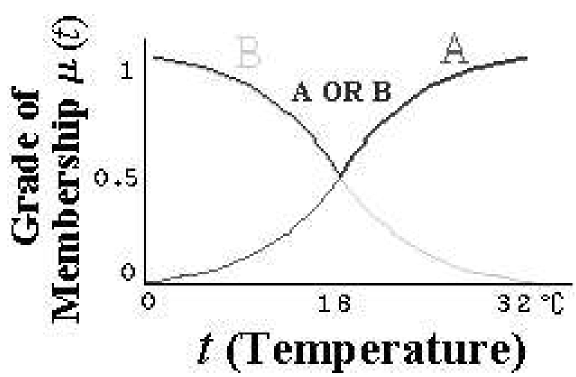
(3) Fuzzy Set Union "OR":

* The fuzzy set of [COLD OR HOT] is a union of A and B fuzzy sets:
$AORB = \mu_A(t) \vee \mu_B(t)(\vee : ma)$



Fuzzy Logic (Cont'd)

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- practical fuzzy applications
  - control system
  - finance
  - pattern recognition
  - expert system
  - database
  - decision making
  - and so on

Basic definitions

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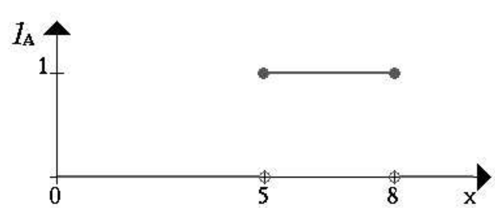
- Crisp sets
  - an example
    * consider a set X of all real numbers between 0 and 10
    * a subset A of X of all real-numbers in the range between 5 and 8
    * characteristic function: assigns a number 1 or 0 to each element in X, depending on whether the element is in the subset A or not



Basic definitions (Cont'd)

Slide 10

- Fuzzy sets
  - an example
    - * describe the set of young people *(B = {set of young people})*
    - * age starts at 0 the lower range of this set ought to be clear
    - * The upper range is rather hard to define (an ex. 20 years)
      →*B = [0,20]*
    - * question ?
      why is somebody on his 20th birthday young and right on the next day not young?
    - * A more natural way to relax the strict separation between young and not young

Basic definitions (Cont'd)

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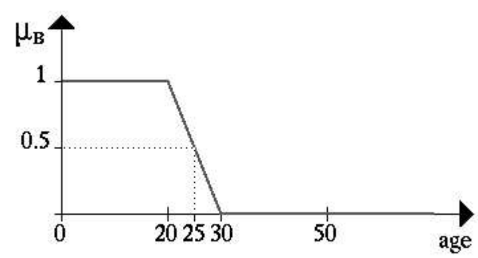
* characteristic function: assigns infinite many alternatives between 0 and 1 (the unit interval I = [0, 1])

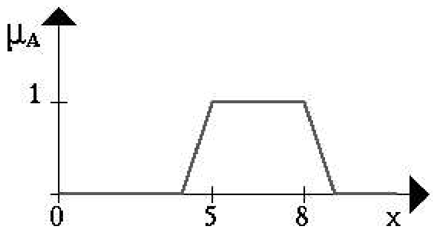


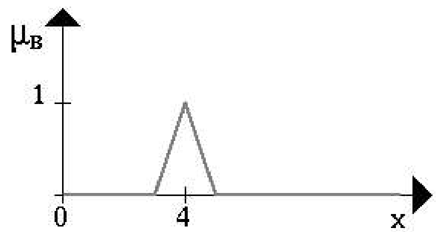
* a 25 years old would still be young to a degree of 50 percent

Basic definitions (Cont'd)

Slide 12

- Operations on Fuzzy Sets
  - two fuzzy sets

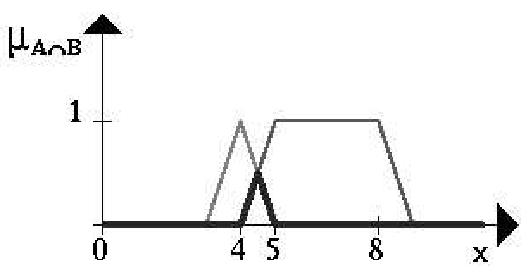




Basic definitions (Cont'd)

Slide 13

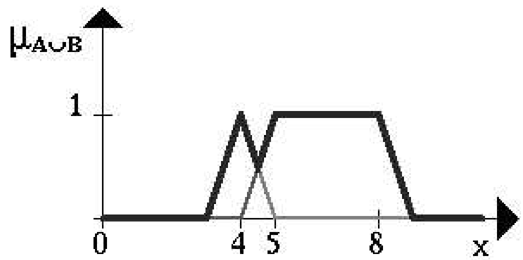
- intersect (Zadeh suggested the minimum operator)



Basic definitions (Cont'd)

Slide 14

– unify (Zadeh suggested the maximum operator)



Basic definitions (Cont'd)

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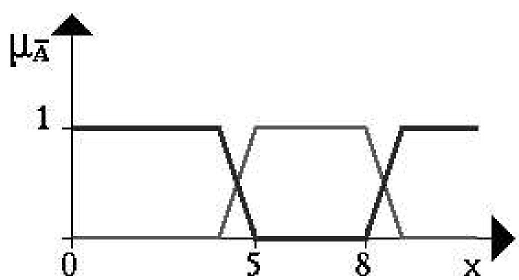
– negate



Fuzzy Mathematics (Ref. book 3)

Slide 16

- Fuzzy sets—basic definitions
  - definition: fuzzy set

If X is a collection of objects denoted generically x then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$

  - where $\mu_{\tilde{A}}(x)$
    - called the membership function or grade of membership of x in $\tilde{A}$
    - maps X to the membership space M
    - when M contains only the two points 0 and 1, $\tilde{A}$ is nonfuzzy
  - ex)
    - $\tilde{A} =$ "*real numbers close to* 10"
      $\rightarrow \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid \mu_{\tilde{A}}(x) = (1 + (x - 10)^2)^{-1}\}$ $\rightarrow \tilde{A} = \int_{\Re} \frac{1}{1+(x-10)^2}/_x$
      $\rightarrow$fig. 2-1
    - another representation
      $\tilde{A} = \mu_{\tilde{A}}(x_1)/_{x_1} + \mu_{\tilde{A}}(x_2)/_{x_2} \cdots = \sum_{i=1}^{n} \mu_{\tilde{A}}(x_i)/_{x_i}$ or $\int_X \mu_{\tilde{A}}(x)/_x$
  - called normal if $sup_x \mu_{\tilde{A}}(x) = 1$ (normalize $\mu_{\tilde{A}}(x)/sup_x \mu_{\tilde{A}}(x)$)
  - definition: support

Fuzzy Mathematics (Ref. book 3) (Cont'd)

Slide 17

The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$

- definition: α-cut

The (crisp) set of elements which belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level-set: $A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha\}$ $A'_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha\}$ is called strong α-level-set or strong α-cut

- definition: convex

A fuzzy set $\tilde{A}$ is convex if $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq min(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)), x_1, x_2 \in X, \lambda \in [0, 1]$

  - fig 2.2 (a) is convex and (b) is non-convex fuzzy set
- definition: subset

A fuzzy set $\tilde{A}$ is called a subset of the fuzzy set $\tilde{B}$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in X$. This is denoted by $\tilde{A} \subseteq \tilde{B}$

- definition: equal

Fuzzy Mathematics (Ref. book 3) (Cont'd)

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> If $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x), \forall x \in X$ then two fuzzy sets $\tilde{A}, \tilde{B}$ are called equal and denoted by $\tilde{A} = \tilde{B}$

- definition: proper subset

> A fuzzy set $\tilde{A}$ is a proper subset of the fuzzy set $\tilde{B}$ if $\tilde{A}$ is a subset of the $\tilde{B}$ and the two sets are not equal (that is, $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x), \forall x \in X$ and $\mu_{\tilde{A}}(x) < \mu_{\tilde{B}}(x), \exists x \in X$. This is denoted by $\tilde{A} \subset \tilde{B}$

- definition: not equal

> If $\mu_{\tilde{A}}(x) \neq \mu_{\tilde{B}}(x), \exists x \in X$, then the two fuzzy sets $\tilde{A}, \tilde{B}$ are said to be "not equal"

- definition: cardinality

> For a finite fuzzy set $\tilde{A}$ the cardinality $| \tilde{A} |$ is defined as $| \tilde{A} | = \sum_{x \in X} \mu_{\tilde{A}}(x)$
> $|| \tilde{A} || = \frac{|\tilde{A}|}{|X|}$ is called the relative cardinality of $\tilde{A}$

Fuzzy Mathematics (Ref. book 3) (Cont'd)

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- Basic set theoretic operations
  - definition: intersection

> The membership function $\mu_{\tilde{C}}(x)$ of the intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined by $\mu_{\tilde{C}}(x) = min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, x \in X$

  - definition: union

> The membership function $\mu_{\tilde{D}}(x)$ of the union $\tilde{D} = \tilde{A} \cup \tilde{B}$ is defined by $\mu_{\tilde{D}}(x) = max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}, x \in X$

  - definition: complement

> The membership function of the complement of a fuzzy set $\tilde{A}$, $\mu_{\complement\tilde{A}}(x)$ is defined by $\mu_{\complement\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)\}, x \in X$

  - other operators for above operations could have been chosen

Extensions

Slide 20

- different extensions of the definition of a fuzzy set
  - human beings can have a crisp image of membership functions in their minds?

    →suggested the notion of a fuzzy set whose membership function itself is a fuzzy set
  - definition: type 2 fuzzy sets

A type 2 fuzzy set is a fuzzy set whose membership values are type 1 fuzzy sets on $[0, 1]$

    where type 1 fuzzy set is a fuzzy set with crisply defined membership functions
  - the operations intersection, union, and complement so far are no longer adequate

Extensions (Cont'd)

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- definition: type m fuzzy sets

A type m fuzzy set is a fuzzy set in X whose membership values are type $m-1$, $m>1$ fuzzy sets on $[0, 1]$
- type m fuzzy sets for large m (even for $m \geq 3$) are hard to deal with and extremely difficult or even impossible to measure them
- other attempts to define fuzziness of type 1 fuzzy sets
  * stochastic fuzzy model by Norwich and Turksen
  * view a fuzzy set as a family of random variables (see Fig. 3-1)

Extensions (Cont'd)

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- different extensions of the operations on fuzzy sets
  - at ordinary fuzzy sets (type 1 fuzzy sets)
  - algebraic operations
    * definition: Cartesian product

      The *Cartesian product* of $\tilde{A}_1, \cdots, \tilde{A}_n$ fuzzy sets is defined in the product space $X_1 \times \cdots \times X_n$ with the membership function $\mu_{(\tilde{A}_1 \times \cdots \times \tilde{A}_n)}(x) = min_i\{\mu_{\tilde{A}_i}(x_i) | x = (x_1, \cdots, x_n), x_i \in X_i\}$
    * definition: power

      The mth power of a fuzzy set $\tilde{A}$ is a fuzzy set with the membership function $\mu_{\tilde{A}^m}(x) = [\mu_{\tilde{A}}(x)]^m, x \in X$
    * definition: algebraic sum

      The *algebraic sum* (probabilistic sum) $\tilde{C} = \tilde{A} + \tilde{B}$ is defined as $\tilde{C} = \{(x, \mu_{\tilde{A}+\tilde{B}}(x)) \mid x \in X\}$ where $\mu_{\tilde{A}+\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$

Extensions (Cont'd)

Slide 23

* definition: bounded sum

  The *bounded sum* $\tilde{C} = \tilde{A} \oplus \tilde{B}$ is defined as $\tilde{C} = \{(x, \mu_{\tilde{A}\oplus\tilde{B}}(x)\}$ where $\mu_{\tilde{A}\oplus\tilde{B}} = min(1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x))$
* definition: bounded difference

  The *bounded difference* $\tilde{C} = \tilde{A} \ominus \tilde{B}$ is defined as $\tilde{C} = \{(x, \mu_{\tilde{A}\ominus\tilde{B}}(x)\}$ where $\mu_{\tilde{A}\ominus\tilde{B}} = max(0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1)$
* definition: algebraic product

  The *algebraic product* of two fuzzy sets $\tilde{C} = \tilde{A} \cdot \tilde{B}$ is defined as $\tilde{C} = \{(x, \mu_{\tilde{A}\cdot\tilde{B}}(x)) \mid x \in X\}$
* See ex. 3-2 (p. 30)

Extensions (Cont'd)

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- set-theoretic operations
  - Hamacher intersection operation

    The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as
    $\tilde{A} \cap \tilde{B} = \{(x, \mu_{\tilde{A}\cap\tilde{B}}(x)) \mid x \in X\}$
    where $\mu_{\tilde{A}\cap\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)}{\gamma+(1-\gamma)(\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x))}, \gamma \geq 0$
  - Hamacher union operation

    The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is defined as
    $\tilde{A} \cup \tilde{B} = \{(x, \mu_{\tilde{A}\cup\tilde{B}}(x)) \mid x \in X\}$
    where $\mu_{\tilde{A}\cup\tilde{B}}(x) = \frac{\mu_{\tilde{A}}(x)+\mu_{\tilde{B}}(x)-(2-\gamma)\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)}{1-(1-\gamma)\mu_{\tilde{A}}(x)\mu_{\tilde{B}}(x)}, \gamma \geq 0$

Extensions (Cont'd)

Slide 25

- t-Norms (triangular-norms) and t-conorms
  - t-norms are two valued functions $\rightarrow t : [0,1] \times [0,1] \rightarrow [0,1]$
  - conditions
    1. $t(0,0) = 0; t(\mu_{\tilde{A}}(x), 1) = t(1, \mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}(x), x \in X$
    2. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$
    3. $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$
    4. $t((\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \mu_{\tilde{C}}(x)) = t(\mu_{\tilde{A}}(x), (\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x)))$
  - t-norms also satisfy the condition
    * $t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$
      with
      $$t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \begin{cases} \mu_{\tilde{A}}(x) & \text{if } \mu_{\tilde{B}}(x) = 1 \\ \mu_{\tilde{B}}(x) & \text{if } \mu_{\tilde{A}}(x) = 1 \\ 0 & \text{else.} \end{cases}$$
    * $t_w(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x))$ is called drastic product or bold intersection
    * min, product, and bounded sum operators belong to the t-norms
  - t-conorms conditions

Extensions (Cont'd)

Slide 26

1. $t(0,0) = 0; t(\mu_{\tilde{A}}(x), 1) = t(1, \mu_{\tilde{A}}(x)) = 1, x \in X$
2. the others are the same as the t-norms
3. max, sum, and bounded difference operators belong to the t-conorms

- appropriate operators for application can be selected considering criteria such as empirical fit, adaptability, numerical efficiency, and so on.

Fuzzy measures and measures of fuzziness

Slide 27

- fuzzy measures and measures of fuzziness are different
- fuzzy measures
  - assigns a value to each crisp set signifying the degree of evidence or belief that a particular element belongs in the set
  - ex.)
    * diagnose an ill patient (an element)
    * belongs in pneumonia, bronchitis, emphysema, common cold (crisp sets) ?
    * assigns a value (degree of evidence) to each crisp set
      →pneumonia (0.45), bronchitis (0.75), emphysema (0), common cold (0.3)
- measures of fuzziness
  - assigns a value to each element signifying its degree of membership in a particular set with unsharp boundaries
  - ex.)
    * young people (a particular set with unsharp boundaries)
    * universal set from 0 to 100 years old people (elements)
    * assigns a value to each element

Fuzzy measures and measures of fuzziness (Cont'd)

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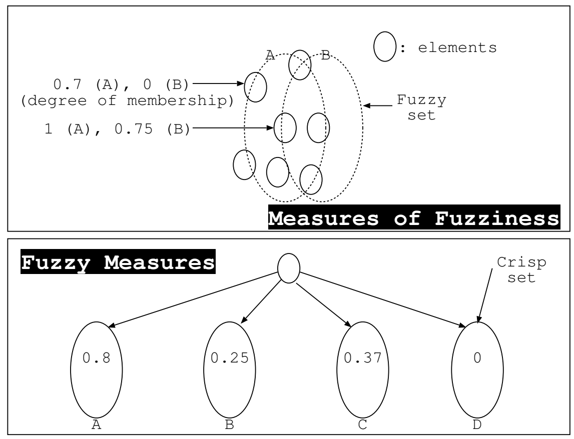
$\rightarrow$10 (1.0), 20 (0.9), 30 (0.5), 40 (0.2), 50 (0)



Fuzzy measures and measures of fuzziness (Cont'd)

Slide 29

- Sugeno defined a fuzzy measure
  - definition: fuzzy measure

A set function g defined on $\mathcal{B}$ which has the following properties is called a fuzzy measure: 1. $g(\phi) = 0, g(X) = 1$ (boundary conditions) 2. If $A, B \in \mathcal{B}$ and $A \subseteq B$ then $g(A) \leq g(B)$ (monotonicity) 3. If $A_n \in \mathcal{B}, A_1 \subseteq A_2 \subseteq \cdots$ then $lim_{n\rightarrow\infty} g(A_n) = g(lim_{n\rightarrow\infty} A_n)$ (continuity)

where $\mathcal{B}$ is Borel field

· $\mathcal{B} \subset \mathcal{P}(X)$

· $\phi \in \mathcal{B}$

· if $A \in \mathcal{B}$, then $\bar{A} \in \mathcal{B}$

· if $\forall i \in \mathcal{N}, A_i \in \mathcal{B}$, then $\bigcup_{i\in\mathcal{N}} A_i \in \mathcal{B}$

Fuzzy measures and measures of fuzziness (Cont'd)

Slide 30

- possibility measure is a special type of the fuzzy measure
  - definition: possibility measure

    A *possibility measure* is a function $\prod : P(X) \rightarrow [0,1]$ with the properties
    1. $\prod(\phi) = 0$
    2. $A \subseteq B \Rightarrow \prod(A) \leq \prod(B)$
    3. $\prod(\cup_{i \in I} A_i) = sup_{i \in I} \prod(A_i)$

    * possibility measure can be built from a possibility distribution
    * possibility distribution : a function $f : X \rightarrow [0,1]$
      $\rightarrow \prod(A) = sup_{x \in A} f(x), A \subset X$
    * f is defined by $f(x) = \prod(\{x\}) \forall x \in X$
    * possibility measure is a fuzzy measure only with two constraints
      1. finite
      2. the possibility distribution is normal (i.e. $sup_{x \in X} \prod(x) = 1$)

Fuzzy measures and measures of fuzziness (Cont'd)

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- other fuzzy measures
  - belief and plausibility measures
  - probability measures
  - possibility and necessity measures
- measures of fuzziness
  - a function: $d : P(X) \rightarrow [0, +\infty]$

    Let $\mu_{\tilde{A}}(x)$ be the membership function of the fuzzy set $\tilde{A}$ for $x \in X$ finite. The measure of fuzziness $d(\tilde{A})$ should have following properties.
    1. $d(\tilde{A}) = 0$ if $\tilde{A}$ is a crisp set in X
    2. $d(\tilde{A})$ assumes a unique maximum if $\mu_{\tilde{A}}(x) = \frac{1}{2}, x \in X$
    3. $d(\tilde{A} \geq d(\tilde{A}')$ if $\tilde{A}'$ is "crisper" than $\tilde{A}$, i.e., if $\mu_{\tilde{A}'}(x) \leq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \leq \frac{1}{2}$ and $\mu_{\tilde{A}'}(x) \geq \mu_{\tilde{A}}(x)$ for $\mu_{\tilde{A}}(x) \geq \frac{1}{2}$
    4. $d(\complement\tilde{A}) = d(\tilde{A})$ where $\complement\tilde{A}$ is the complement of $\tilde{A}$

  - De Luca and Termini suggested as a measure of fuzziness the "entropy" of a fuzzy set

Fuzzy measures and measures of fuzziness (Cont'd)

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– definition: measure of fuzziness

> The entropy as a measure of fuzzy set $\tilde{A} = \{(x, \mu_A(x)\}$ is defined as $d(\tilde{A}) = H(\tilde{A}) + H(\complement\tilde{A}), x \in X$ $H(\tilde{A}) = -K\sum_{i=1}^{n} \mu_{\tilde{A}}(x_i) ln(\mu_{\tilde{A}}(x_i))$ where n is the number of elements in the support of $\tilde{A}$ and K is a positive constant

* using Shannon's function $S(x) = -xlnx - (1-x)ln(1-x)$, simplify as

  > $d(\tilde{A}) = K\sum_{i=1}^{n} S(\mu_{\tilde{A}}(x_i))$

* ex.) show example 4-1 (p. 42)

Fuzzy measures and measures of fuzziness (Cont'd)

Slide 33

– Yager's measure of fuzziness

* distance between a fuzzy set and its complement

  $D_p(\tilde{A}, \complement\tilde{A}) = \left[\sum_{i=1}^{n} \mid \mu_{\tilde{A}}(x_i) - \mu_{\complement\tilde{A}}(x_i) \mid^p\right]^{\frac{1}{p}}, p = 1, 2, 3, \cdots$

* definition: measure of fuzziness

  > A measure of the fuzziness of $\tilde{A}$ can be defined as $f_p(\tilde{A}) = 1 - \frac{D_p(\tilde{A}, \complement\tilde{A})}{||supp(\tilde{A})||}$

* for $p = 1$, $D_p(\tilde{A}, \complement\tilde{A})$ yields the Hamming metric

  $D_1(\tilde{A}, \complement\tilde{A}) = \sum_{i=1}^{n} \mid \mu_{\tilde{A}}(x_i) - \mu_{\complement\tilde{A}}(x_i) \mid$

* for $p = 2$, we arrive at the Euclidean metric

  $D_2(\tilde{A}, \complement\tilde{A}) = \left(\sum_{i=1}^{n} (\mu_{\tilde{A}}(x_i) - \mu_{\complement\tilde{A}}(x_i))^2\right)^{\frac{1}{2}}$

* see ex. 4-2 (p. 43)

The Extension Principle and Applications

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- definition: extension principle

Let X be a Cartesian product of universes $X = X_1, \cdots, X_r$, and $\tilde{A}_1, \cdots, \tilde{A}_r$ be r fuzzy sets in $X_1, \cdots, X_r$, respectively, f is a mapping from X to a universe $Y, y = f(x_1, \cdots, x_r)$, Then a fuzzy set $\tilde{B}$ in Y is defined by $\tilde{B} = \{(y, \mu_{\tilde{B}}(y) \mid f(x_1, \cdots, x_r), (x_1, \cdots, x_r) \in X\}$ where

$$\mu_{\tilde{B}}(y) = \begin{cases} sup_{(x_1, \cdots, x_r) \in f^{-1}(y)} min\{\mu_{\tilde{A}_1}(x_1), \cdots, \mu_{\tilde{A}_r}(x_r)\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

- For $r = 1$ the extension principle reduces to

$f(\tilde{A}) = f\{(x, \mu(x)) \mid x \in X\}$ where

$$\mu_{(\tilde{A})}(y) = \begin{cases} sup_{x \in f^{-1}(y) \mu_{\tilde{A}}(x)}, & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

- see ex. 5-1 (p. 48)

The Extension Principle and Applications (Cont'd)

Slide 35

- operations for type 2 fuzzy sets
  - extension principle can be used to define set theoretic operations for type 2 fuzzy sets
  - let two fuzzy sets of type 2 be $\tilde{A}(x) = \{(x, \mu_{\tilde{A}}(x)\}, \tilde{B}(x) = \{(x, \mu_{\tilde{B}}(x)\}$ where $\mu_{\tilde{A}}(x) = \{(u_i, \mu_{u_i}(x)) \mid x \in X, u_i, \mu_{u_i}(x) \in [0, 1]\}$ and $\mu_{\tilde{B}}(x) = \{(v_i, \mu_{v_i}(x)) \mid x \in X, v_i, \mu_{v_i}(x) \in [0, 1]\}$
  - the u_i and v_j are degrees of membership of type 1 fuzzy sets and $\mu_{u_i}(x)$ and $\mu_{v_j}(x)$ their membership functions

The Extension Principle and Applications (Cont'd)

Slide 36

- definition: union in type 2 fuzzy set

  Let two fuzzy sets of type 2 be defined as above. The membership function of their union is then defined by:

$$\begin{aligned}\mu_{\tilde{A}\cup\tilde{B}}(x) &= \mu_{\tilde{A}}(x) \cup \mu_{\tilde{B}}(x) \\ &= \{(w, \mu_{\tilde{A}\cup\tilde{B}}(w) \mid w = max\{u_i, v_j\}, u_i, v_j \in [0,1]\}\end{aligned}$$

  where $\mu_{\tilde{A}\cup\tilde{B}}(w) = sup_{w=max\{u_i,v_j\}} min\{\mu_{u_i}(x), \mu_{v_j}(x)\}$

- definition: intersection

$$\begin{aligned}\mu_{\tilde{A}\cap\tilde{B}}(x) &= \mu_{\tilde{A}}(x) \cap \mu_{\tilde{B}}(x) \\ &= \{(w, \mu_{\tilde{A}\cap\tilde{B}}(w) \mid w = min\{u_i, v_j\}, u_i, v_j \in [0,1]\}\end{aligned}$$

  where $\mu_{\tilde{A}\cap\tilde{B}}(w) = sup_{w=min\{u_i,v_j\}} min\{\mu_{u_i}(x), \mu_{v_j}(x)\}$

- see ex. 5-2 (p. 50)

The Extension Principle and Applications (Cont'd)

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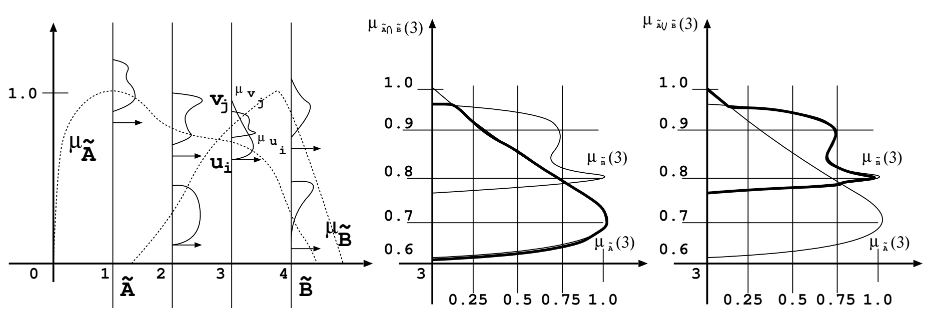


The Extension Principle and Applications (Cont'd)

Slide 38

- Algebraic operators with fuzzy numbers
  - definition: fuzzy number

    A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R such that
    1. It exists exactly one $x_0 \in R, \mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$)
    2. $\mu_{\tilde{M}}(x)$ is piecewise continuous
  - definition: positive and negative

    A fuzzy number $\tilde{M}$ is called positive (negative) if its membership function is such that $\mu_{\tilde{M}}(x) = 0, \forall x < 0 (\forall x > 0)$

The Extension Principle and Applications (Cont'd)

Slide 39

- definition: binary operation

  Let $F(R)$ be the set of real fuzzy numbers and $\tilde{M}, \tilde{N} \in F(R)$ with $\mu_{\tilde{N}}(x)$ and $\mu_{\tilde{M}}(x)$ continuous membership functions, then by application of the extension principle for the binary operation $* : R \times R \to R$: $\mu_{\tilde{M} \circledast \tilde{N}}(z) = sup_{z=x \times y} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y))$

  * properties of the extended operation $\circledast$
    1. For any commutative operation $*$ the extended operation $\circledast$ is also commutative
    2. For any associative operation $*$ the extended operation $\circledast$ is also associative
- definition: unary operation

  For unary operations $f : X \to Y, X = X_1$ the extension principle reduces to $\mu_{f(\tilde{M})}(z) = sup_{x \in f^{-1}(z)} \mu_{\tilde{M}}(x)$

The Extension Principle and Applications (Cont'd)

Slide 40

– properties of the extended addition

1. $\ominus(\tilde{M} \oplus \tilde{N}) = (\ominus\tilde{M}) \oplus (\ominus\tilde{N})$
2. $\oplus$ is commutative
3. $\oplus$ is associative
4. $0 \in R \subseteq F(R)$ is the neutral element for $\oplus$, that is, $\tilde{M} + 0 = \tilde{M}, \forall \tilde{M} \in F(R)$
5. For $\oplus$ there does not exist an inverse element, that is, $\forall \tilde{M} \in F(R)$
   $R : \tilde{M} \oplus -\tilde{M} \neq 0 \in R$

– properties of $\odot$

1. $(-\tilde{M}) \odot \tilde{N} = -(\tilde{M} \odot \tilde{N})$
2. $\odot$ is commutative
3. $\odot$ is associative
4. $\tilde{M} \odot 1 = \tilde{M}, 1 \in R \subseteq F$ (R is the neutral element for $\odot$, that is
   $\tilde{M} \odot 1 = \tilde{M}, \forall \tilde{M} \in F(R)$)
5. For $\odot$ there does not exist an inverse element, that is, $\forall \tilde{M} \in F(R)$
   $R : \tilde{M} \odot -\tilde{M}^{-1} \neq 1$

The Extension Principle and Applications (Cont'd)

Slide 41

– extended subtraction

* If $f(x) = -x$, then $\ominus\tilde{M} = \{(x, \mu_{\ominus\tilde{M}}(x)) \mid x \in X\}$ where $\mu_{\ominus\tilde{M}(x)} = \mu_{\tilde{M}}(-x)$

$$\begin{aligned} \mu_{\tilde{M}\ominus\tilde{N}}(z) &= sup_{z=x-y} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \\ &= sup_{z=x+y} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(-y)) \\ &= sup_{z=x+y} min(\mu_{\tilde{M}}(x), \mu_{-\tilde{N}}(y)) \end{aligned}$$

– extended division

$$\begin{aligned} \mu_{\tilde{M}\oslash\tilde{N}}(z) &= sup_{z=x/y} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(y)) \\ &= sup_{z=xy} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}}(1/y)) \\ &= sup_{z=xy} min(\mu_{\tilde{M}}(x), \mu_{\tilde{N}^{-1}}(y)) \end{aligned}$$

The Extension Principle and Applications (Cont'd)

Slide 42

- example
  - $\tilde{M} = \{(2;1),(3;0.5)\}$ and $\tilde{N} = \{(3;1),(4;0.5)\}$
  - $\tilde{M} \oplus \tilde{N} = \{(5;1),(6;0.5),(7;0.5)\}$
  - $\tilde{M} \ominus \tilde{N} = \{(-2;0.5),(-1;1),(0;0.5)\}$
  - $\tilde{M} \otimes \tilde{N} = \{(6;1),(8;0.5),(9;0.5),(12;0.5)\}$
  - $\tilde{M} \oslash \tilde{N} = \{(2/3;1),(2/4;0.5),(3/4;0.5),(1;0.5)\}$

- LR-representation of fuzzy sets
  - reference function f of a fuzzy number iff
    1. $f(-x) = f(x)$
    2. $f(0) = 1$
    3. f is decreasing on $[0, +\infty]$
  - definition: LR-representation

The Extension Principle and Applications (Cont'd)

Slide 43

A fuzzy number $\tilde{M}$ is of LR-type if there exist reference functions L (for left) , R (for right) and scalars $\alpha > 0, \beta > 0$ with m, called the mean value of $\tilde{M}$, is a real number and α and β are called the left and right spreads, respectively. Symbolically $\tilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$.

$$\mu_{\tilde{M}}(x) = \begin{cases} L(\frac{m-x}{\alpha}) & \text{for } x \leq m; \\ R(\frac{x-m}{\beta}) & \text{for } x \geq m; \end{cases}$$

- ex.) Let $L(x) = \frac{1}{1+x^2}$, $R(x) = \frac{1}{1+2|x|}$, and $\alpha = 2, \beta = 3, m = 5$, then

$$\mu_{\tilde{5}}(x) = \begin{cases} L(\frac{5-x}{2}) = \frac{1}{1+(\frac{5-x}{2})^2} & \text{for } x \leq 5; \\ R(\frac{x-5}{3}) = \frac{1}{1+|\frac{2(x-5)}{3}|} & \text{for } x \geq 5; \end{cases}$$

The Extension Principle and Applications (Cont'd)

Slide 44

- Theorem
  - Let $\tilde{M}, \tilde{N}$ be two fuzzy numbers of LR-type:
    $\tilde{M} = (m, \alpha, \beta)_{LR}, \tilde{N} = (n, \gamma, \delta)_{LR}$
  - then
    1. $(m, \alpha, \beta)_{LR} \oplus (n, \gamma, \delta)_{LR} = (m + n, \alpha + \gamma, \beta + \delta)_{LR}$
    2. $-(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{RL}$
    3. $(m, \alpha, \beta)_{LR} \ominus (n, \gamma, \delta)_{RL} = (m - n, \alpha + \delta, \beta + \gamma)_{LR}$
  - product and devision
    - $(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, m\gamma + n\alpha, m\delta + n\beta)_{LR}$ for $\tilde{M}, \tilde{N}$ positive
    - $(m, \alpha, \beta)_{RL} \odot (n, \gamma, \delta)_{LR} \approx (mn, n\alpha - m\delta, n\beta - m\gamma)_{RL}$ for $\tilde{N}$ positive, $\tilde{M}$ negative
    - $(m, \alpha, \beta)_{LR} \odot (n, \gamma, \delta)_{LR} \approx (mn, -n\beta - m\delta, n\alpha - m\gamma)_{RL}$ for $\tilde{M}, \tilde{N}$ negative

Fuzzy Relations and Fuzzy Graphs

Slide 45

- fuzzy relations are fuzzy subsets of $X \times Y$, that is, mapping from $X \to Y$.
- definition: fuzzy relation

Let $X, Y \subseteq R$ be universal sets then $\tilde{R} = \{(x, y), \mu_{\tilde{R}}(x, y)) \mid (x, y) \subseteq X \times Y\}$ is called a fuzzy relation on $X \times Y$

- ex. 6-1)
  - Let $X = Y = R$ and $\tilde{R} :=$ "considerably larger than."
  - The membership function of the fuzzy relation

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ \frac{(x-y)}{10y} & \text{for } y < x \leq 11y \\ 1 & \text{for } x > 11y \end{cases}$$

  - another membership function could be

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0 & \text{for } x \leq y \\ (1 + (y - x)^{-2})^{-1} & \text{for } x > y \end{cases}$$

Fuzzy Relations and Fuzzy Graphs (Cont'd)

  - ∗ for discrete supports fuzzy relations can also be defined by matrices (see ex 6-2. p 62)

Slide 46

- definition: generalized fuzzy relation

  Let $X, Y \subseteq R$ and $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X\}$ $\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) \mid y \in Y\}$ two fuzzy sets. Then $\tilde{R} = \{[(x, y), \mu_{\tilde{R}}(x, y)], (x, y) \in X \times Y\}$ is a fuzzy relation on $\tilde{A}$ and $\tilde{B}$ if $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{A}}(x), \forall(x, y) \in X \times Y$ and $\mu_{\tilde{R}}(x, y) \leq \mu_{\tilde{B}}(y), \forall(x, y) \in X \times Y$

  - fuzzy relations are fuzzy sets in product spaces
  - therefore set theoretic and algebraic operations can be defined
  - definition: union

    Let $\tilde{R}$ and $\tilde{Z}$ be two fuzzy relations in the same product space; The *union*(intersection) of $\tilde{R}$ with $\tilde{Z}$ is then defined by
    $\mu_{\tilde{R}\cup\tilde{Z}}(x, y) = max\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, (x, y) \in X \times Y$
    $\mu_{\tilde{R}\cap\tilde{Z}}(x, y) = min\{\mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y)\}, (x, y) \in X \times Y$
  - see ex. 6-3 (p. 63)

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 47

- definition: projection

  Let $\tilde{R} = \{(x, y), \mu_{\tilde{R}}(x, y), (x, y) \in X \times Y\}$ be a fuzzy binary relation. The first projection of $\tilde{R}$ is then defined as $\tilde{R}^{(1)} = \{(x, max_y \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$ The second projection is defined as $\tilde{R}^{(2)} = \{(y, max_x \mu_{\tilde{R}}(x, y)) \mid (x, y) \in X \times Y\}$ and the total projection as $\tilde{R}^{(T)} = max_x max_y \{\mu_{\tilde{R}}(x, y)), (x, y) \in X \times Y\}$

  - see ex. 6-4 (p. 64)
  - called a "shadow" and can be extended to general space: $X = X_1 \times \cdots \times X_n$.
  - distinct fuzzy relations in the same universe can have the same projection
  - there must be uniquely defined largest relation for each projection
    →called *cylindrical extension of the projection relation*

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 48

- definition: cylindrical extension

$\tilde{R}_{qL} \subseteq X$ is the largest relation in X the projection of which is $\tilde{R}_q$. $\tilde{R}_{qL}$ is then called the cylindrical extension of $\tilde{R}_q$ and $\tilde{R}_q$ the base of $\tilde{R}_{qL}$

- see ex. 6-5 (p. 65)
- definition: join and meet

Let $\tilde{R}$ be a fuzzy relation on $X = X_1 \times \cdots \times X_n$ and $\tilde{R}_1$ and $\tilde{R}_2$ be two fuzzy projections on $X_1 \times \cdots \times X_r$ and $X_s \times \cdots \times X_n$, respectively, with $s \leq r+1$ and $\tilde{R_{1L}}, \tilde{R_{2L}}$ their respective cylindrical extensions. The join of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as $\tilde{R_{1L}} \cap \tilde{R_{2L}}$ and their meet as $\tilde{R_{1L}} \cup \tilde{R_{2L}}$

Fuzzy Relations and Fuzzy Graphs (Cont'd)

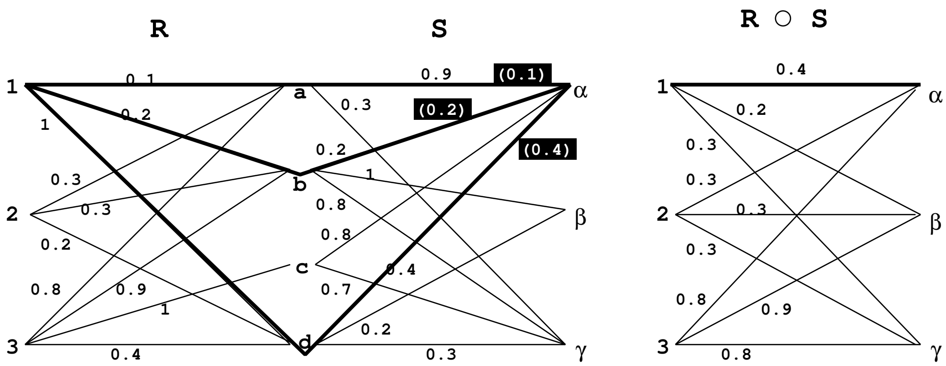
Slide 49

- composition of fuzzy relations
  - fuzzy relations in different product spaces can be combined by the operation "composition"
  - max-min-composition is the best known and the most frequently used one
  - max-product or max-average compositions shows appealing results
  - definition: max-min composition

Let $\tilde{R}_1(x,y), (x,y) \in X \times Y$ and $\tilde{R}_2(y,z) \in Y \times Z$ be two fuzzy relations. The max-min composition $\tilde{R}_1$ max-min $\tilde{R}_2$ is then the fuzzy set: $\tilde{R}_1 \circ \tilde{R}_2 = \{[(x,z), max_y\{min\{\mu_{\tilde{R}_1}(x,y), \mu_{\tilde{R}_2}(y,z)\}\}] \mid x \in X, y \in Y, z \in Z\}$

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 50



Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 51

- definition: max-*-composition

  Let $\tilde{R}_1$ and $\tilde{R}_2$ be defined as in above definition. The max-*-composition of $\tilde{R}_1$ and $\tilde{R}_2$ is then defined as: $\tilde{R}_1 \circledast \tilde{R}_2 = \{[(x,z), max_y(\mu_{\tilde{R}_1}(x,y) * \mu_{\tilde{R}_2}(y,z))] \mid x \in X, y \in Y, z \in Z\}$

  * If * is an associative operation which is monotonously non-decreasing
  * then, the max-*-composition corresponds the max-min-composition
  * two special cases of the max-*-composition
  * definition: max-prod and max-av-compositions

    Let $\tilde{R}_1$ and $\tilde{R}_2$ be defined as in definition join and meet. The max-prod-composition $\tilde{R}_1 \overset{\cdot}{\circ} \tilde{R}_2$ and the max-av-composition $\tilde{R}_1 \overset{av}{\circ} \tilde{R}_2$ are then defined as follows: $\tilde{R}_1 \overset{\cdot}{\circ} \tilde{R}_2(x,z) = \{[(x,z), max_y(\mu_{\tilde{R}}(x,y) \cdot \mu_{\tilde{R}}(y,z))], x \in X, y \in Y, z \in Z\}$ $\tilde{R}_1 \overset{av}{\circ} \tilde{R}_2(x,z) = \{[(x,z), 1/2max(\mu_{\tilde{R}_1}(x,y) + \mu_{\tilde{R}_2}(y,z))], x \in X, y \in Y, z \in Z\}$

- see ex. 6-6 (p. 67)

Fuzzy Relations and Fuzzy Graphs (Cont'd)

- binary relations on a single set
  - relation $\tilde{R}$ is denoted by $\tilde{R}(X, X) \subseteq X \times X$
  - can be expressed by matrix, sagittal diagram, simple diagram, and table
  - ex.)

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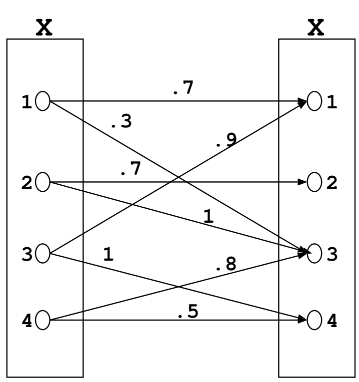
표 1: relation matrix

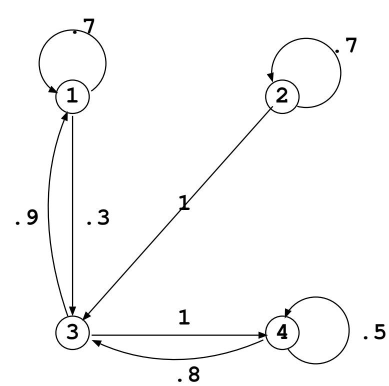
		X			
		1	2	3	4
X	1	.7	0	.3	0
	2	0	.7	1	0
	3	.9	0	0	1
	4	0	0	.8	.5

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 53



Sagittal diagram



Simple diagram

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 54

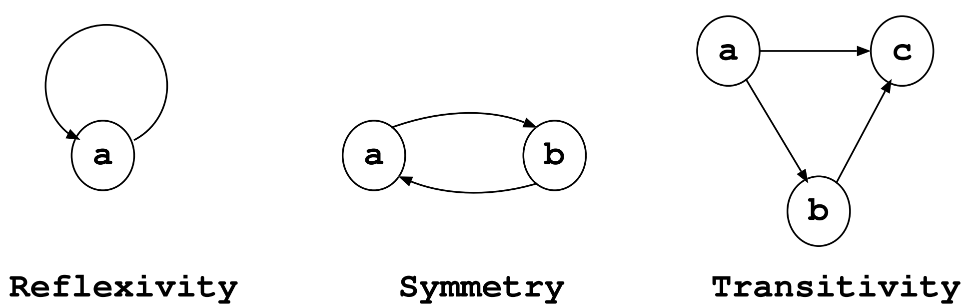
표 2: relation table

x	y	$\mu_{\tilde{R}}(x, y)$
1	1	.7
1	3	.3
2	2	.7
2	3	1
3	1	.9
3	4	1
4	3	.8
4	4	.5

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 55

– characteristic properties: reflexivity, symmetry and transitivity



Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 56

- properties of the max-min-composition
  - associativity
    - $(\tilde{R}_3 \circ \tilde{R}_2) \circ \tilde{R}_1 = \tilde{R}_3 \circ (\tilde{R}_2 \circ \tilde{R}_1)$
    - hence $\tilde{R}_1 \circ \tilde{R}_1 \circ \tilde{R}_1 = \tilde{R}_1^3$ the 3rd power of a fuzzy relation is defined
  - reflexitivity
    - definition: reflexitivity

Let $\tilde{R}$ be a fuzzy relation in $X \times X$
1. $\tilde{R}$ is called reflexive if $\mu_{\tilde{R}}(x,x) = 1, \forall x \in X$
2. $\tilde{R}$ is called ε-reflexive if $\mu_{\tilde{R}}(x,x) = \varepsilon, \forall x \in X$
3. $\tilde{R}$ is called weakly reflexive if

$$\left.\begin{array}{l} \mu_{\tilde{R}}(x,y) \le \mu_{\tilde{R}}(x,x) \\ \mu_{\tilde{R}}(y,x) \le \mu_{\tilde{R}}(x,x) \end{array}\right\} \forall x,y \in X$$

    - see ex. 6-7 (p. 70)
    - If $\tilde{R}_1$ and $\tilde{R}_2$ are reflexive fuzzy relations then max-min-composition $\tilde{R}_1 \circ \tilde{R}_2$ is

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 57

also reflexive

  - symmetry and antisymmetric

A fuzzy relation $\tilde{R}$ is called symmetric if $R(x,y) = R(y,x)$ A relation is called antisymmetric if for

$$x \neq y \quad \left.\begin{array}{l} \text{either}\,\mu_{\tilde{R}}(x,y) \neq \mu_{\tilde{R}}(y,x) \\ \text{or}\,\mu_{\tilde{R}}(x,y) = \mu_{\tilde{R}}(y,x) = 0 \end{array}\right\} \forall x,y \in X$$

  - perfectly antisymmetric

if for $x \neq y$ whenever $\mu_{\tilde{F}}(x,y) > 0$ then $\mu_{\tilde{R}}(y,x) = 0, \forall x,y \in X$

  - see ex. 6-8 (p. 71)
  - transitivity

A fuzzy relation $\tilde{R}$ is called (max-min) transitive if $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$

  - see ex. 6-10 (p. 73)

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 58

- special fuzzy relations
  - definition: similarity relation

A similarity relation is a fuzzy relation $\mu_s(\cdot)$ which is reflexive, symmetrical, and max-min-transitive

  - see ex. 6-14 (p. 77)
  - similarity relation of a finite number of elements can also be represented by a "similarity-tree" (see p. 78)
  - definition: fuzzy preorder relation

A fuzzy relation which is (max-min)-transitive and reflexive is called a fuzzy preorder relation

  - definition: fuzzy order relation

A fuzzy relation which is (min-max)-transitive, reflexive, and anti-symmetric is called a fuzzy order relation

Fuzzy Relations and Fuzzy Graphs (Cont'd)

Slide 59

- definition: perfect fuzzy order relation

A fuzzy relation which is (min-max)-transitive, reflexive, and perfectly anti-symmetric is called a perfect fuzzy order relation, also called fuzzy partial order relation

- definition: total fuzzy order relation (or fuzzy linear ordering)

A total fuzzy order relation or a fuzzy linear ordering is a fuzzy order relation such that $\forall x, y \in X;\ x \neq y$ either $\mu_{\tilde{R}}(x,y) > 0$ or $\mu_{\tilde{R}}(y,x) > 0$.

Possibility Theory vs. Probability Theory

Slide 60

- the comparison between probability theory and fuzzy set theory is difficult with two reasons
  - different levels
  - fuzzy set theory is no longer a uniquely defined mathematical structure
- definition: fuzzy restriction

Let $\tilde{F}$ be a fuzzy set of the universe U characterized by a membership function $\mu_{\tilde{F}}(u)$. $\tilde{F}$ is a fuzzy restriction on the variable X. the assignment of the values u to X has the form: $X = u : \mu_{\tilde{F}}(u)$

  - If $\tilde{R}(X) = \tilde{F}$, then $\tilde{R}(X)$ is called *relational assignment equation*
  - Let $A(X)$ is an implied attribute of the variable X, then $\tilde{R}(A(X)) = \tilde{F}$

Possibility Theory vs. Probability Theory (Cont'd)

Slide 61

- ex.)
  * X: Jack
  * $A(X)$: age of Jack
  * $\tilde{F}$: fuzzy set "young"
  * restriction of "(the age of) Jack is young"
    →$\tilde{R}(A(X)) = \tilde{F} \Longrightarrow \tilde{R}$(Age(Jack)) = young
    →If $u = 28$ (Jack's age is 28) and $\mu_{\tilde{F}}(u) = 0.7$
    →this means the degree of possibility that Jack is 28 given the proposition "Jack is young" is 0.7

Possibility Theory vs. Probability Theory (Cont'd)

Slide 62

- possibility distribution

  Let $\tilde{F}$ be a fuzzy set in a universe U with its membership function $\mu_{\tilde{F}}(u)$ and let X be a variable in U and $\tilde{F}$ act as a fuzzy restriction, $\tilde{R}(X)$. Then the proposition "X is $\tilde{F}$" ($\tilde{R}(X) = \tilde{F}$) associates a possibility distribution, π_X. The possibility distribution function, $\pi_X(u)$ is defined to be numerically equal to the membership function $\mu_{\tilde{F}}(u)$.

  - see ex. 8-2 and 8-3 (p. 107)

Fuzzy logic and approximate reasoning

Slide 63

- Linguistic variables
  - variables whose values are not numbers but words or sentences in a natural or artificial language
  - definition: linguistic variable

    A linguistic variable is characterized by a quintuple $(x, T(x), U, G, \tilde{M})$ in which x is the name of the variable; $T(x)$ denotes the term-set of x with U universe of discourse; G is syntactic rule; and $\tilde{M}$ is a semantic rule that assigns a meaning
  - see ex. 9-1 (p. 122)

Fuzzy logic and approximate reasoning (Cont'd)

Slide 64

- two linguistic variables, "Truth" and "Probability"
  - * "Probability" see fig. 9-2 (p. 125)
  - * Baldwin defines the "Truth" as $\mu_{\text{very true}}(v) = (\mu_{\text{true}}(v))^2, v \in [0,1]$
    $\mu_{\text{fairly true}}(v) = (\mu_{\text{true}}(v))^{\frac{1}{2}}, v \in [0,1]$
    - · see fig. 9-3 (p. 126)
  - * Zadeh suggests the "Truth" as

$$\mu_{\text{true}}(v) = \begin{cases} 0 & \text{for } 0 \le v \le a \\ 2(\frac{v-a}{1-a})^2 & \text{for } a \le v \le \frac{a+1}{2} \\ 1 - (\frac{v-1}{1-a})^2 & \text{for } \frac{a+1}{2} \le v \le 1 \end{cases}$$

    - · where $v = (1+a)/2$ is called the *crossover point* and $a \in [0,1]$
    - · the membership function of "false": $\mu_{\text{false}} = \mu_{\text{true}}(1-v), 0 \le v \le 1$
    - · see fig. 9-4 (p. 127)
    - · T(Truth) = {truth, not true, very true, not very true, $\cdots$, false, not false, very false, $\cdots$, not very true and not very false, $\cdots$}

Fuzzy logic and approximate reasoning (Cont'd)

Slide 65

- definition: modifier

A linguistic hedge or a modifier is an operation, which modifies the meaning of a term or more general of a fuzzy set. If $\tilde{A}$ is a fuzzy set then the modifier m generates the (composite) term $\tilde{B} = m(\tilde{A})$

  - * frequently used modifiers
    - · Concentration: $\mu_{Con(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^2$
    - · Dilation: $\mu_{Dil(\tilde{A})}(u) = (\mu_{\tilde{A}}(u))^{\frac{1}{2}}$
    - · Contrast intensification:

$$\mu_{Int(\tilde{A})}(u) = \begin{cases} 2(\mu_{\tilde{A}}(u))^2 & \text{for } \mu_{\tilde{A}}(u) \in [0, 0.5] \\ 1 - 2(1-\mu_{\tilde{A}}(u))^2 & \text{otherwise} \end{cases}$$

  - * If $\tilde{A}$ is a term (a fuzzy set) then
    - · very $\tilde{A}$ = Con($\tilde{A}$)
    - · more or less $\tilde{A}$ = Dil($\tilde{A}$)
    - · plus $\tilde{A} = \tilde{A}^{1.25}$
    - · slightly $\tilde{A}$ = Int[plus $\tilde{A}$ and not (very $\tilde{A}$)]

Fuzzy logic and approximate reasoning (Cont'd)

Slide 66

- fuzzy logic
  - in Boolean logic
    - truth values can be 0(false) or 1(true)
    - 2^4 operators
    - tautologies
      - modus ponens: $(A \wedge (A \Rightarrow B)) \Rightarrow B$
      - modus tollens: $((A \Rightarrow B) \wedge \neg B) \Rightarrow \neg A$
      - syllogism: $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$
      - contraposition: $(A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)$
  - fuzzy logic
    - is an extension of set theoretic multi-valued logic in which the truth values are linguistic variables
  - truth value of proposition
    - if a proposition "u is A" has $v(A) \in [0, 1]$ truth value
    - then the truth value of $not\ A$ is given by $v(not\ A) = 1 - v(A)$
    - definition: the truth value of $not\ A$

Fuzzy logic and approximate reasoning (Cont'd)

Slide 67

> If $\tilde{v}(A)$ is a normalized fuzzy set, $\tilde{V}(A) = \{(v_i, \mu_i) \mid i = 1, \cdots, n, v_i \in [0, 1]\}$ then by applying the extension principle, the truth value of $\tilde{v}(not\ A)$ is defined as $\tilde{v}(not\ A) = \{(1 - v_i, \mu_i) \mid i = 1, \cdots, n, v_i \in [0, 1]\}$

  - truth tables and linguistic approximation
    - in fuzzy logic the number of truth values is infinite →operators are not possible
    - use linguistic variable we can tabulate
    - Zadeh suggests truth tables using four-valued logic with "true", "false", "undecided", and "unknown"
    - "unknown" is interpreted as "true or false" and "undecided" is denoted by $\ominus$

Fuzzy logic and approximate reasoning (Cont'd)

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표 3: Truth table for "and"

$\wedge$	T	F	$T+F$
T	T	F	$T+F$
F	F	F	F
$T+F$	$T+F$	F	$T+F$

표 4: Truth table for "or"

$\vee$	T	F	$T+F$
T	T	T	T
F	T	F	$T+F$
$T+F$	T	$T+F$	T

Fuzzy logic and approximate reasoning (Cont'd)

Slide 69

표 5: Truth table for "not"

	$\neg$
T	F
F	T
$T+F$	$T+F$

Fuzzy logic and approximate reasoning (Cont'd)

Slide 70

∗ Baldwin version

true $= \{(v, \mu_{true}(v) = v \mid v \in [0,1]\}$
false $= \{(v, \mu_{false}(v) = 1 - \mu_{true}(v) \mid v \in [0,1]\}$
very true $= \{(v, (\mu_{true}(v))^2) \mid v \in [0,1]\}$
fairly true $= \{(v, (\mu_{true}(v))^{\frac{1}{2}}) \mid v \in [0,1]\}$
undecided $= \{(v, 1) \mid v \in [0,1]\}$

absolutely true $= \{(v, \mu_{AT}(v) \mid v \in [0,1]\}$ with $\mu_{AT}(v) = \begin{cases} 1 & \text{for } v = 1 \\ 0 & \text{otherwise} \end{cases}$

absolutely false $= \{(v, \mu_{AF}(v) \mid v \in [0,1]\}$ with $\mu_{AF}(v) = \begin{cases} 1 & \text{for } v = 0 \\ 0 & \text{otherwise} \end{cases}$

$(very)^k$ true →absolutely true as $k \to \infty$
$(very)^k$ false →absolutely false as $k \to \infty$
$(fairly)^k$ true →undecided as $k \to \infty$
$(fairly)^k$ false →undecided as $k \to \infty$
see fig. 9-3 (p. 136)

Fuzzy logic and approximate reasoning (Cont'd)

Slide 71

- approximate reasoning
  - modus ponens: $(A \wedge (A \Rightarrow B)) \Rightarrow B$

Premise	A is true
Implication	If A then B
Conclusion	B is true

  - definition: generalized modus ponens by Zadeh

Let $\tilde{A}, \tilde{A}', \tilde{B}, \tilde{B}'$ be fuzzy statements, then the generalized modus ponens reads:

Premise	x is A'
Implication	If x is A then y is B
Conclusion	y is B'

  - definition: Zadeh's compositional rule

Fuzzy logic and approximate reasoning (Cont'd)

Slide 72

Let $\tilde{R}(x), \tilde{R}(x,y)$ and $\tilde{R}(y), x \in X, y \in Y$ be fuzzy relations in $X, X \times Y$ and Y, respectively. Let $\tilde{A}$ and $\tilde{B}$ denote particular fuzzy sets in X and $X \times Y$. Then the compositional rule of inference asserts, that the solution of the relational assignment equations $\tilde{R}(x) = \tilde{A}$ and $\tilde{R}(x,y) = \tilde{B}$ is given by $\tilde{R}(y) = \tilde{A} \circ \tilde{B}$, where $\tilde{A} \circ \tilde{B}$ is the composition of $\tilde{A}$ and $\tilde{B}$

∗ see ex. 9-8 (p. 138)

An application (Fuzzy Control)

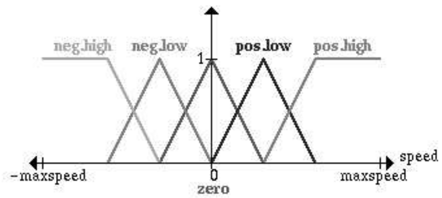
Slide 73

- the most important applications of fuzzy theory
- an example (Inverted pendulum)
  - balance a pole on a mobile platform that can move in only two directions
  - knowledge can be expressed in a very natural way using linguistic variables
    - ∗ negative high
    - ∗ negative low
    - ∗ zero
    - ∗ positive low
    - ∗ positive high

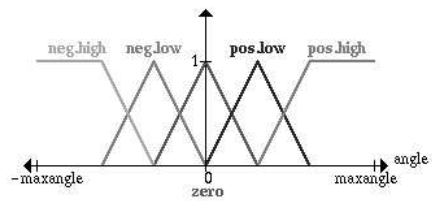
An application (Fuzzy Control) (Cont'd)

Slide 74

- membership functions
  - speed



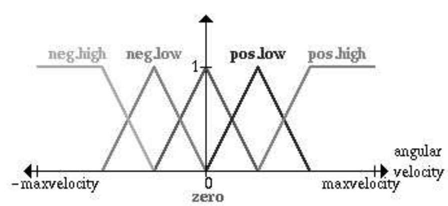
  - angle



An application (Fuzzy Control) (Cont'd)

Slide 75

  - velocity



- several rules that say what to do in certain situations
  - situation : consider pole is in upright position with low velocity in positive direction
  - action : compensate the pole's movement by moving the platform in the same direction at low speed
  - formalized form
    If angle is zero and angular velocity is zero then speed shall be zero.
    If angle is zero and angular velocity is pos. low then speed shall be pos. low.

An application (Fuzzy Control) (Cont'd)

Slide 76

∗ summarize all applicable rules in a table

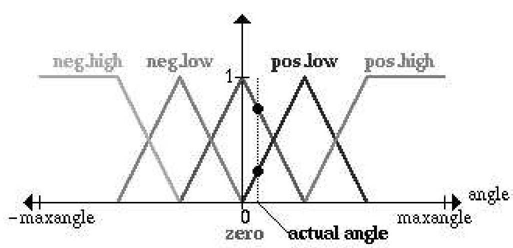
		angle				
	speed	NH	NL	Z	PL	PH
v	NH			NH		
e	NL			NL	Z	
l	Z	NH	NL	Z	PL	PH
o	PL		Z	PL		
c	PH			PH		

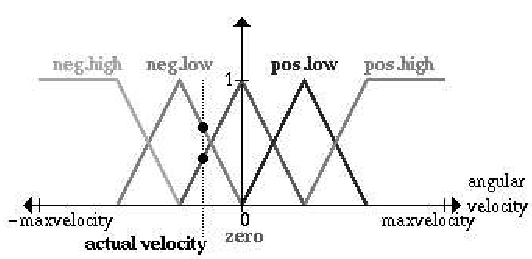
An application (Fuzzy Control) (Cont'd)

Slide 77

– a control example

∗ two values for angle and angular velocity

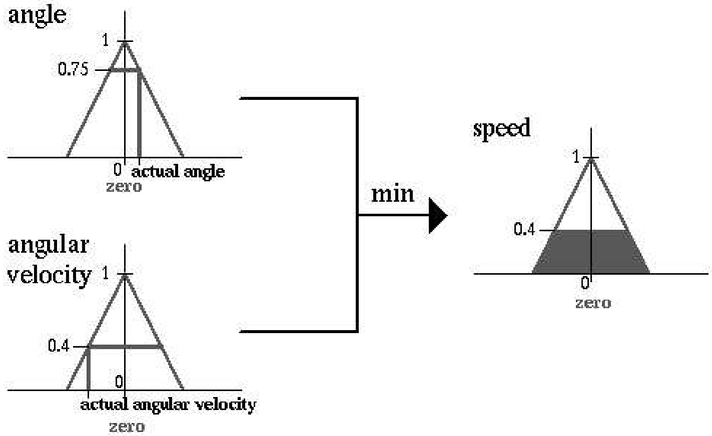




An application (Fuzzy Control) (Cont'd)

Slide 78

∗ apply a rule *(If angle is zero and angular velocity is zero then speed is zero)*



Since the two parts of the condition of our rule are connected by an AND we calculate min(0.75,0.4) = 0.4 and cut the fuzzy set "zero" of the variable "speed" at this level (according to our rule).

An application (Fuzzy Control) (Cont'd)

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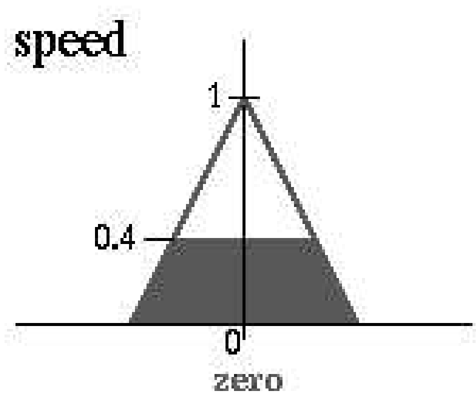
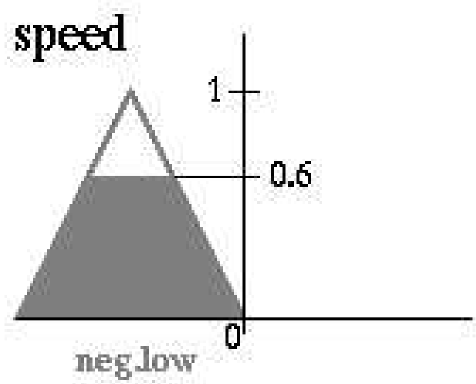


An application (Fuzzy Control) (Cont'd)

Slide 80

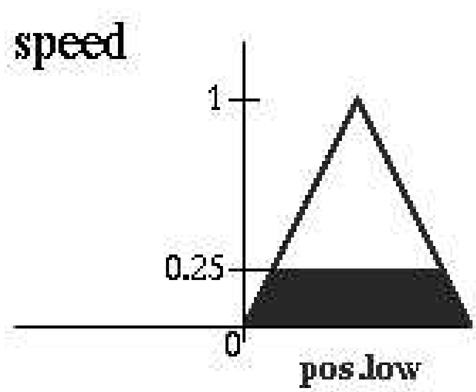
* apply a rule *(If angle is zero and angular velocity is negative low then speed is negative low)*



An application (Fuzzy Control) (Cont'd)

Slide 81

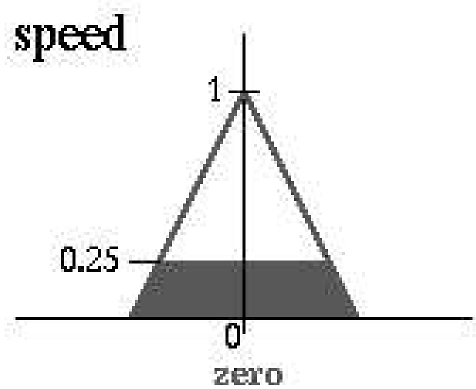
* apply a rule *(If angle is positive low and angular velocity is zero then speed is positive low)*



An application (Fuzzy Control) (Cont'd)

Slide 82

* apply a rule *(If angle is positive low and angular velocity is negative low then speed is zero)*



An application (Fuzzy Control) (Cont'd)

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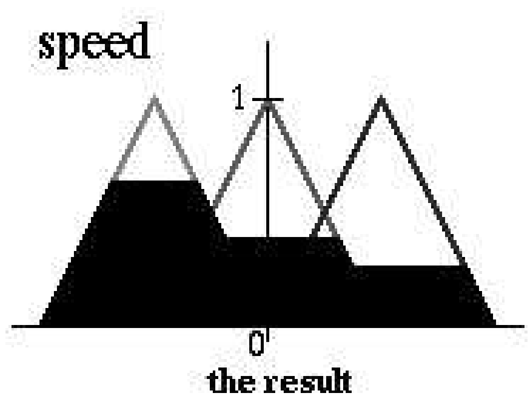
* the four results yield the overall result

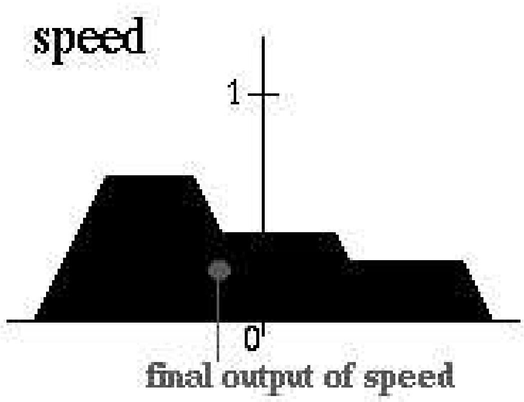


* the result is a fuzzy set (of speed)
* so we have to choose one representative value as the final output
* there are several heuristic methods (defuzzification methods), e.g. the center of

An application (Fuzzy Control) (Cont'd)

Slide 84

gravity of the fuzzy set:



An application (Fuzzy Control) (Cont'd)

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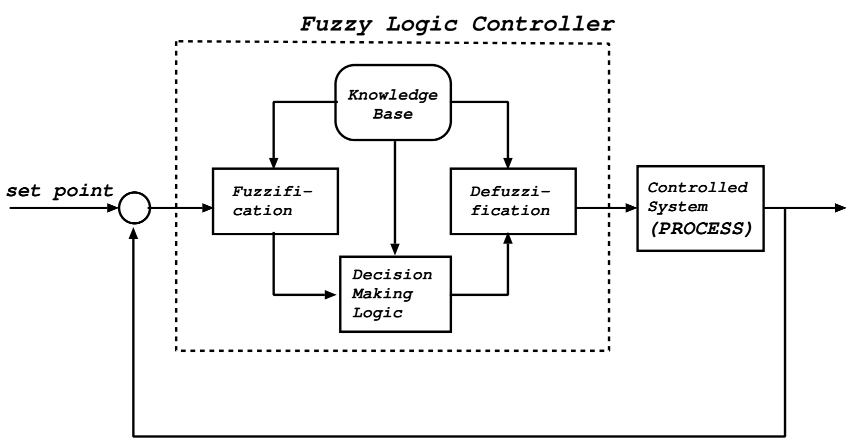
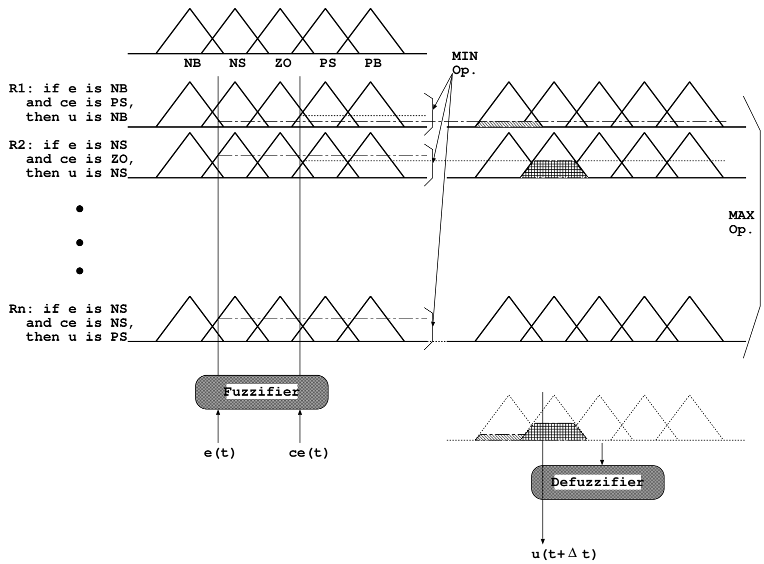
- overall control structure



An application (Fuzzy Control) (Cont'd)

Slide 86

- overall control operation



An application (Fuzzy Control) (Cont'd)

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- application areas
  - good areas
    * for very complex processes, when there is no simple mathematical model
    * for highly nonlinear processes
    * if the processing of (linguistically formulated) expert knowledge is to be performed
  - bad areas
    * conventional control theory yields a satisfying result
    * an easily solvable and adequate mathematical model already exists
    * the problem is not solvable